5-4 Medians and Altitudes

Common Core State Standards

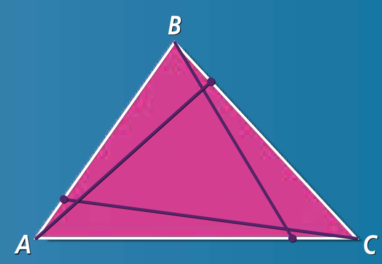
G-CO.C.10 Prove theorems about triangles . . . the medians of a triangle meet at a point. **Also G-SRT.B.5**

MP 1, MP 3, MP 5, MP 7, MP 8

Objective To identify properties of medians and altitudes of a triangle

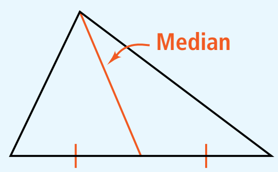
Solve It! Getting Ready!

Draw a large acute scalene $\triangle ABC$. On each side, mark the point that is $\frac{1}{5}$ of the distance from one of the side's endpoints, as shown in the diagram. Connect each of these points to the opposite vertex. Repeat this process for $\frac{1}{4}$ and $\frac{1}{3}$. What do you think the result will be for $\frac{1}{2}$? Check your answer. Were you correct?



You can use different colors for the sets of segments so you can see the pattern more easily.

MATHEMATICAL PRACTICES In the Solve It, the last set of segments you drew are the triangle's medians. A **median of a triangle** is a segment whose endpoints are a vertex and the midpoint of the opposite side.



Essential Understanding A triangle's three medians are always concurrent.

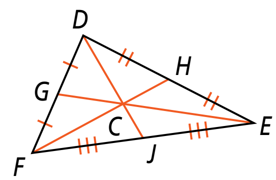
Lesson Vocabulary
- median of a triangle
- centroid of a triangle
- altitude of a triangle
- orthocenter of a triangle

take note

Theorem 5-8 Concurrency of Medians Theorem

The medians of a triangle are concurrent at a point that is two thirds the distance from each vertex to the midpoint of the opposite side.

$DC = \frac{2}{3}DJ$ $\qquad EC = \frac{2}{3}EG$ $\qquad FC = \frac{2}{3}FH$



You will prove Theorem 5-8 in Lesson 6-9.

In a triangle, the point of concurrency of the medians is the **centroid of the triangle.** The point is also called the *center of gravity* of a triangle because it is the point where a triangular shape will balance. For any triangle, the centroid is always inside the triangle.

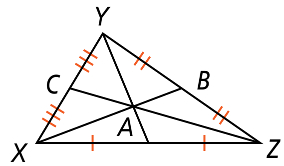
Problem 1 Finding the Length of a Median

GRIDDED RESPONSE

Plan

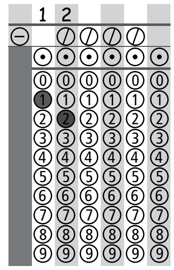
How do you use the centroid?
Write an equation relating the length of the whole median to the length of the segment from the vertex to the centroid.

In the diagram at the right, $XA = 8$. What is the length of $\overline{XB}$?



A is the centroid of $\triangle XYZ$ because it is the point of concurrency of the triangle's medians.

$XA = \frac{2}{3}XB$	Concurrency of Medians Theorem
$8 = \frac{2}{3}XB$	Substitute 8 for XA.
$\left(\frac{3}{2}\right)8 = \left(\frac{3}{2}\right)\frac{2}{3}XB$	Multiply each side by $\frac{3}{2}$.
$12 = XB$	Simplify.



Got It? 1. a. In the diagram for Problem 1, $ZA = 9$. What is the length of $\overline{ZC}$?

b. Reasoning What is the ratio of ZA to AC? Explain.

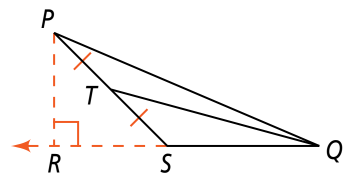
An **altitude of a triangle** is the perpendicular segment from a vertex of the triangle to the line containing the opposite side. An altitude of a triangle can be inside or outside the triangle, or it can be a side of the triangle.

Problem 2 Identifying Medians and Altitudes

Plan

How do you determine whether a segment is an altitude or a median?
Look at whether the segment is perpendicular to a side (altitude) and/or bisects a side (median).

A For $\triangle PQS$, is $\overline{PR}$ a *median*, an *altitude*, or *neither*? Explain.



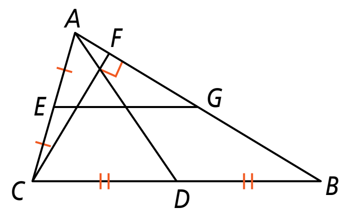
$\overline{PR}$ is a segment that extends from vertex P to the line containing $\overline{SQ}$, the side opposite P. $\overline{PR} \perp \overrightarrow{QR}$, so $\overline{PR}$ is an altitude of $\triangle PQS$.

B For $\triangle PQS$, is $\overline{QT}$ a *median*, an *altitude*, or *neither*? Explain.

$\overline{QT}$ is a segment that extends from vertex Q to the side opposite Q. Since $\overline{PT} \cong \overline{TS}$, T is the midpoint of $\overline{PS}$. So $\overline{QT}$ is a median of $\triangle PQS$.

Got It? 2. For $\triangle ABC$, is each segment a *median*, an *altitude*, or *neither*? Explain.

a. $\overline{AD}$ **b.** $\overline{EG}$ **c.** $\overline{CF}$



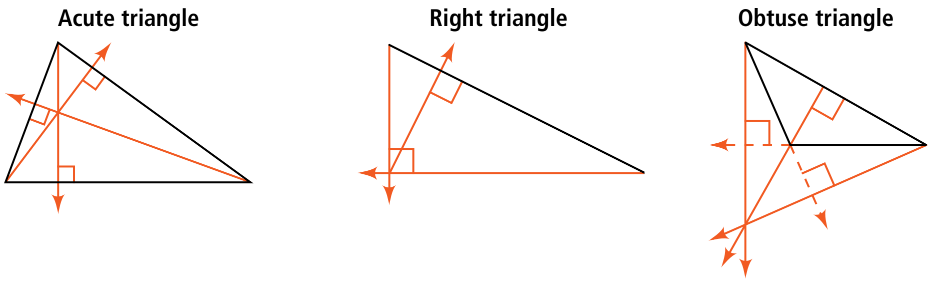
take note

Theorem 5-9 Concurrency of Altitudes Theorem

The lines that contain the altitudes of a triangle are concurrent.

You will prove Theorem 5-9 in Lesson 6-9.

The lines that contain the altitudes of a triangle are concurrent at the **orthocenter of the triangle.** The orthocenter of a triangle can be inside, on, or outside the triangle.



Problem 3 Finding the Orthocenter

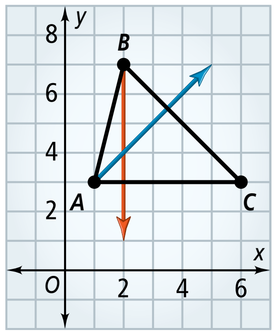
$\triangle ABC$ has vertices $A(1, 3)$, $B(2, 7)$, and $C(6, 3)$. What are the coordinates of the orthocenter of $\triangle ABC$?

Know	Need	Plan
The coordinates of the three vertices	The intersection point of the triangle's altitudes	Write the equations of the lines that contain two of the altitudes. Then solve the system of equations.

Think

Which two altitudes should you choose? It does not matter, but the altitude to $\overline{AC}$ is a vertical line, so its equation will be easy to find.

Step 1 Find the equation of the line containing the altitude to $\overline{AC}$. Since $\overline{AC}$ is horizontal, the line containing the altitude to $\overline{AC}$ is vertical. The line passes through the vertex $B(2, 7)$. The equation of the line is $x = 2$.



Step 2 Find the equation of the line containing the altitude to $\overline{BC}$. The slope of the line containing $\overline{BC}$ is $\frac{3-7}{6-2} = -1$. Since the product of the slopes of two perpendicular lines is -1, the line containing the altitude to $\overline{BC}$ has slope 1.

The line passes through the vertex $A(1, 3)$. The equation of the line is $y - 3 = 1(x - 1)$, which simplifies to $y = x + 2$.

Step 3 Find the orthocenter by solving this system of equations: $x = 2$, $y = x + 2$

$y = 2 + 2$ Substitute 2 for x in the second equation.

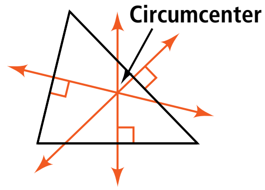
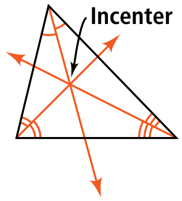
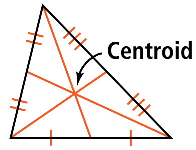
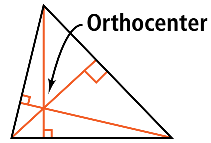
$y = 4$ Simplify.

The coordinates of the orthocenter are (2, 4).

Got It? 3. $\triangle DEF$ has vertices $D(1, 2)$, $E(1, 6)$, and $F(4, 2)$. What are the coordinates of the orthocenter of $\triangle DEF$?

take note

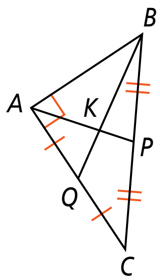
Concept Summary   Special Segments and Lines in Triangles

Perpendicular Bisectors	Angle Bisectors	Medians	Altitudes
			

Lesson Check

Do you know HOW?

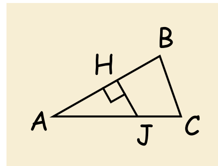
Use $\triangle ABC$ for Exercises 1–4.



1. Is $\overline{AP}$ a *median* or an *altitude*?
2. If $AP = 18$, what is KP?
3. If $BK = 15$, what is KQ?
4. Which two segments are altitudes?

Do you UNDERSTAND? MATHEMATICAL PRACTICES

5. **Error Analysis** Your classmate says she drew $\overline{HJ}$ as an altitude of $\triangle ABC$. What error did she make?



6. **Reasoning** Does it matter which two altitudes you use to locate the orthocenter of a triangle? Explain.
7. **Reasoning** The orthocenter of $\triangle ABC$ lies at vertex A. What can you conclude about $\overline{BA}$ and $\overline{AC}$? Explain.

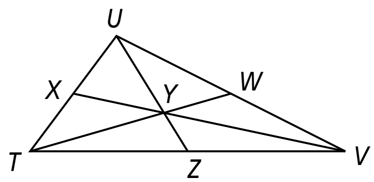
Practice and Problem-Solving Exercises MATHEMATICAL PRACTICES

A Practice

In $\triangle TUV$, Y is the centroid.

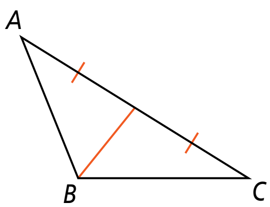
See Problem 1.

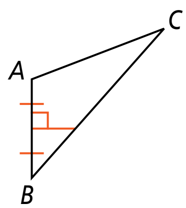
8. If $YW = 9$, find TY and TW.
9. If $YU = 9$, find ZY and ZU.
10. If $VX = 9$, find VY and YX.

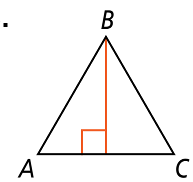


For $\triangle ABC$, is the red segment a *median*, an *altitude*, or *neither*? Explain.

See Problem 2.

11. 

12. 

13. 

Coordinate Geometry **Find the coordinates of the orthocenter of $\triangle ABC$.**

See Problem 3.

14. $A(0, 0)$
$B(4, 0)$
$C(4, 2)$

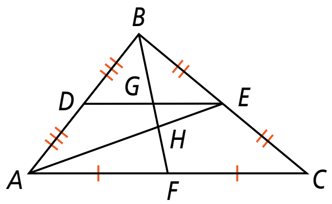
15. $A(2, 6)$
$B(8, 6)$
$C(6, 2)$

16. $A(0, -2)$
$B(4, -2)$
$C(-2, -8)$

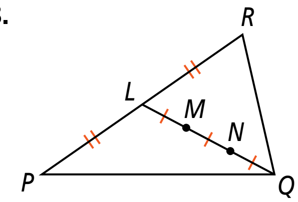
B Apply

Name the centroid.

17.

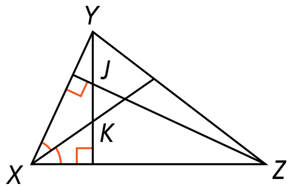


18.

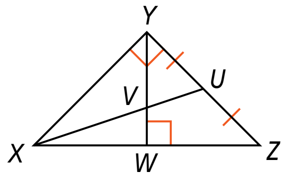


Name the orthocenter of $\triangle XYZ$.

19.

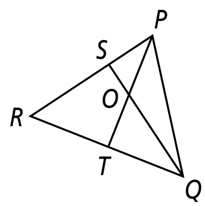


20.



21. Think About a Plan In the diagram at the right, $\overline{QS}$ and $\overline{PT}$ are altitudes and $m\angle R = 55$. What is $m\angle POQ$?

- What does it mean for a segment to be an altitude?
- What do you know about the sum of the angle measures in a triangle?
- How do you sketch overlapping triangles separately?



Constructions **Draw a triangle that fits the given description. Then construct the centroid and the orthocenter.**

22. acute scalene triangle, $\triangle LMN$

23. obtuse isosceles triangle, $\triangle RST$

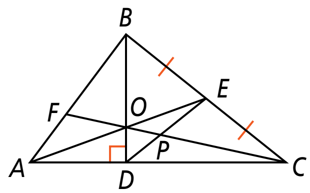
In Exercises 24–27, name each segment.

24. a median in $\triangle ABC$

25. an altitude in $\triangle ABC$

26. a median in $\triangle BDC$

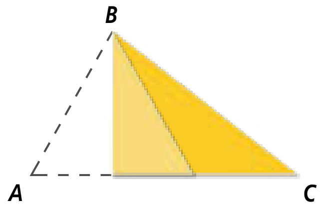
27. an altitude in $\triangle AOC$



28. Reasoning A centroid separates a median into two segments. What is the ratio of the length of the shorter segment to the length of the longer segment?

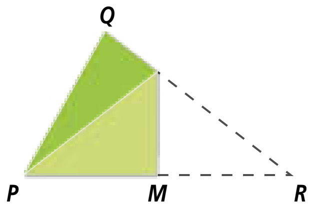
Paper Folding **The figures below show how to construct altitudes and medians by paper folding. Refer to them for Exercises 29 and 30.**

Folding an Altitude

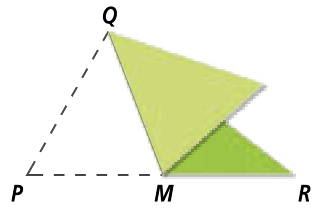


Fold the triangle so that a side $\overline{AC}$ overlaps itself and the fold contains the opposite vertex B.

Folding a Median



Fold one vertex R to another vertex P. This locates the midpoint M of a side.

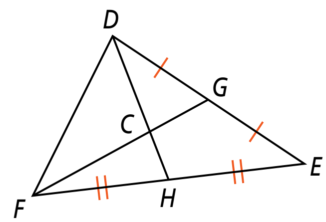


Unfold the triangle. Then fold it so that the fold contains the midpoint M and the opposite vertex Q.

29. Cut out a large triangle. Fold the paper carefully to construct the three medians of the triangle and demonstrate the Concurrency of Medians Theorem. Use a ruler to measure the length of each median and the distance of each vertex from the centroid.

30. Cut out a large acute triangle. Fold the paper carefully to construct the three altitudes of the triangle and demonstrate the Concurrency of Altitudes Theorem.

31. In the figure at the right, C is the centroid of $\triangle DEF$. If $GF = 12x^2 + 6y$, which expression represents CF?

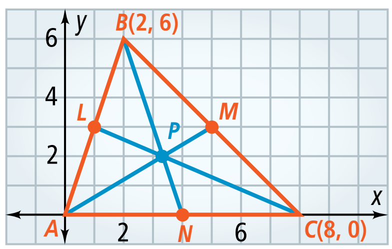


Ⓐ $6x^2 + 3y$
Ⓑ $4x^2 + 2y$
Ⓒ $8x^2 + 4y$
Ⓓ $8x^2 + 3y$

32. Reasoning What type of triangle has its orthocenter on the exterior of the triangle? Draw a sketch to support your answer.

33. Writing Explain why the median to the base of an isosceles triangle is also an altitude.

34. Coordinate Geometry $\triangle ABC$ has vertices $A(0, 0)$, $B(2, 6)$, and $C(8, 0)$. Complete the following steps to verify the Concurrency of Medians Theorem for $\triangle ABC$.



a. Find the coordinates of midpoints L, M, and N.
b. Find equations of $\overleftrightarrow{AM}$, $\overleftrightarrow{BN}$, and $\overleftrightarrow{CL}$.
c. Find the coordinates of P, the intersection of $\overleftrightarrow{AM}$ and $\overleftrightarrow{BN}$. This point is the centroid.
d. Show that point P is on $\overleftrightarrow{CL}$.
e. Use the Distance Formula to show that point P is two thirds of the distance from each vertex to the midpoint of the opposite side.

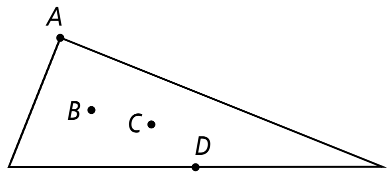
Challenge

35. Constructions A, B, and O are three noncollinear points. Construct point C such that O is the orthocenter of $\triangle ABC$. Describe your method.

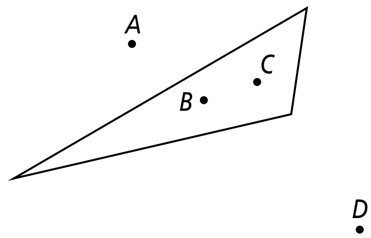
36. Reasoning In an isosceles triangle, show that the circumcenter, incenter, centroid, and orthocenter can be four different points, but all four must be collinear.

A, B, C, and D are points of concurrency for the triangle. Determine whether each point is a *circumcenter*, *incenter*, *centroid*, or *orthocenter*. Explain.

37.



38.

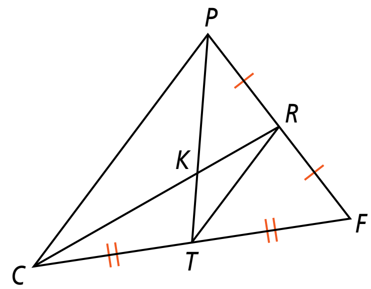


39. **History** In 1765, Leonhard Euler proved that, for any triangle, three of the four points of concurrency are collinear. The line that contains these three points is known as Euler's Line. Use Exercises 37 and 38 to determine which point of concurrency does not necessarily lie on Euler's Line.

Standardized Test Prep

SAT/ACT

For Exercises 40 and 41, use the figure at the right.



40. If $CR = 24$, what is KR?

(A) 6 (C) 12

(B) 8 (D) 16

41. If $TR = 12$ what is CP?

(F) 16 (H) 24

(G) 18 (I) 36

Extended Response

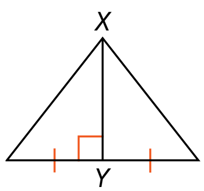
42. The orthocenter of a triangle lies outside the triangle. Where are its circumcenter, incenter, and centroid located in relation to the triangle? Draw and label diagrams to support your answers.

Mixed Review

Is $\overline{XY}$ a *perpendicular bisector*, an *angle bisector*, or *neither*? Explain.

See Lesson 5-3.

43.



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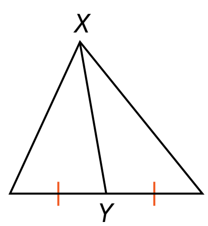

Get Ready! **To prepare for Lesson 5-5, do Exercises 45–47.**

Write the negation of each statement.

See Lesson 2-2.

45. Two angles are congruent.

46. You are not 16 years old.

47. $m\angle A < 90$